Aug 2004

Day 1, Q10 part b)

(this is a supplement to the soln in the book, demonstrating the integration steps to build the matrix M)

Solution. We are looking for solutions to the variational formulation to the given ODE. This leads to the equation

$$\sum_{j=1}^{N}\left[\int_0^1 \phi_i'(x)\phi_j'(x)(1+x)dx + \int_0^1 \phi_i(x)\phi_j(x)dx\right]u_j = \int_0^1 \phi_i(x)(x^2-x)dx$$

which is of the form

$$Mu = f$$

for an $N \times N$ stiffness matrix M and $N \times 1$ load vector f. The actual evaluations come below.

First, the diagonal terms:

$$\begin{aligned}
M_{11} &= \int_0^{1/3}(-3)^2(1+x)dx + \int_0^{1/3}(1-3x)^2dx \\
&= \frac{65}{18} \\
M_{22} &= \int_0^{1/3}(3)^2(1+x)dx + \int_{1/3}^{2/3}(-3)^2(1+x)dx + \int_0^{1/3}(3x)^2dx + \int_{1/3}^{2/3}(2-3x)^2dx \\
&= \frac{74}{9} \\
M_{33} &= \int_{1/3}^{2/3}(3)^2(1+x)dx + \int_{2/3}^{1}(-3)^2(1+x)dx + \int_{1/3}^{2/3}(3x-1)^2dx + \int_{2/3}^{1}(3-3x)^2 \\
&= \frac{92}{9} \\
M_{44} &= \int_{2/3}^{1}(3)^2(1+x)dx + \int_{2/3}^{1}(3x-2)^2dx \\
&= \frac{101}{18}
\end{aligned}$$

Now, the off-diagonal terms. Notice that the hat functions, as drawn in the book, have no overlap for the following matrix elements $1,3$, $1,4$, $2,4$. Therefore, we have

$$\begin{aligned}
M_{13} &= M_{31} = 0 \\
M_{14} &= M_{41} = 0 \\
M_{24} &= M_{42} = 0
\end{aligned}$$

For the remaining non-zero terms we have

$$\begin{aligned}
M_{12} = M_{21} &= \int_0^{1/3} (-3)(3)(1+x)dx + \int_0^{1/3} (1-3x)(3x)dx \\
&= -\frac{31}{9} \\
M_{23} = M_{32} &= \int_{1/3}^{2/3} (-3)(3)(1+x)dx + \int_{1/3}^{2/3} (2-3x)(3x-1)dx \\
&= -\frac{40}{9} \\
M_{34} = M_{43} &= \int_{2/3}^{1} (-3)(3)(1+x)dx + \int_{2/3}^{1} (3-3x)(3x-2)dx \\
&= -\frac{47}{9}
\end{aligned}$$

So, the stiffness matrix with nodes at $x = 0, 1/3, 2/3$, and 1 becomes

$$M = \begin{pmatrix} \frac{65}{18} & -\frac{31}{9} & 0 & 0 \\ -\frac{31}{9} & \frac{74}{9} & -\frac{40}{9} & 0 \\ 0 & -\frac{40}{9} & \frac{92}{9} & -\frac{47}{9} \\ 0 & 0 & -\frac{47}{9} & \frac{101}{18} \end{pmatrix}$$